

On algebraic structures in supersymmetric principal chiral model

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Abstract

Using the Poisson current algebra of the supersymmetric principal chiral model, we develop the algebraic canonical structure of the model by evaluating the fundamental Poisson bracket of the Lax matrices that fits into the $r-s$ matrix formalism of nonultralocal integrable models. The fundamental Poisson bracket has been used to compute the Poisson bracket algebra of the monodromy matrix that gives the conserved quantities in involution.

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1 Introduction

In some recent investigations [1], [2], the classical integrability of supersymmetric principal chiral model (SPCM) has been studied. It has been shown in [2], that for the SPCM there exist two families of local conserved quantities in involution, each with finitely many members whose spins are exactly the exponents of the underlying Lie algebra of the model, with no repetition modulo the Coxeter number. Along with the existence of local conserved quantities in SPCM, there exist non-local conserved quantities as well [1]-[7]. Furthermore, it has been shown in [1] that the SPCM also admits a one-parameter family of transformation on superfields leading to a superfield Lax formalism and a zero-curvature representation. The superfield Lax formalism is shown to be related to the super Backlund transformation and the super Riccati equations of the SPCM.

In integrable field theories some integrable canonical structures are associated with the Lax pair. The Lax pair in general is a pair of matrices, which are functions of fields and a spectral parameter. The matrices obey a Poisson bracket algebra which in many cases is ultralocal i.e. the algebra does not contain the derivatives of the delta function. Such models are referred to as ultralocal models. The ultralocality leads to a Poisson bracket algebra of monodromy matrix and the Jacobi identity gives the classical Yang-Baxter equation for an r -matrix[11]-[23]. In fact, the Yang-Baxter equation leads to the existence of commuting conserved quantities, ensuring the integrability of the model[11]-[23].

The r -matrix method has also been employed to non-ultralocal models i.e. the models for which the algebra of Lax matrices contains derivatives of the delta function [11] – [23]. Some examples of such integrable models are the principal chiral model (PCM), the complex sine-Gordon theory (CSG), the Wess-Zumino-Witten model (WZW), $O(N)$ sigma model, etc. The $r - s$, matrix approach has been adopted to study such non-ultralocal models in which the r -matrices are no longer anti-symmetric and may also depend on dynamical variables giving an extended dynamical Yang-Baxter equation.[17].

The purpose of this paper is to study the integrability of the supersymmetric principal chiral model (SPCM) as a non-ultralocal model, using the $r - s$ matrix approach of the Poisson bracket algebra of monodromy matrix. We demonstrate that the SPCM which is known to be integrable provide an explicit

realization of the $r - s$ matrix formalism developed for bosonic integrable models by Maillet [20], [21]. Starting with a Lax formalism, we develop a canonical $r - s$ matrix approach for the SPCM and obtain the Poisson bracket algebra of the monodromy matrix in terms of the $r - s$ matrices for which the consistency condition implies an extended non-dynamical Yang-Baxter equation.

2 The SPCM and its Poisson bracket algebra

Following [1], [2], we define the supersymmetric principal chiral model as follows. Let us consider a superfield $G(x, \theta)$ with values in a Lie group \mathcal{G} . This superfield $G(x, \theta)$ is a function of the space coordinates x^\pm and anti-commuting coordinates θ^\pm ³. The superspace Lagrangian of the SPCM is then given as

$$\mathcal{L} = \frac{1}{2} \text{Tr} (D_+ G^{-1} D_- G), \quad (2.1)$$

where

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i\theta^\pm \partial_\pm \quad (2.2)$$

are the superspace covariant derivatives and

$$G(x, \theta) G^{-1}(x, \theta) = 1 = G^{-1}(x, \theta) G(x, \theta). \quad (2.3)$$

The superspace Lagrangian \mathcal{L} is invariant under the following transformation

$$\mathcal{G}_L \times \mathcal{G}_R : \quad G(x^\pm, \theta^\pm) = \mathcal{U} G(x^\pm, \theta^\pm) \mathcal{V}^{-1}, \quad (2.4)$$

where \mathcal{U} and \mathcal{V} are \mathcal{G}_L and \mathcal{G}_R valued matrix superfields respectively. The Noether conserved superfield currents associated with the global transformation are

$$J_\pm^L = iD_\pm G G^{-1}, \quad J_\pm^R = -iG^{-1} D_\pm G, \quad (2.5)$$

where $J_\pm^{R,L}$ are the Grassmann odd and are Lie algebra valued, i.e., $J_\pm = J_\pm^a T^a$, where $\{T^a\}$ is the set of generators of the Lie algebra \mathfrak{g} of the Lie group \mathcal{G} ⁴. The superfield equation of motion of the SPCM is the superfield conservation equation

$$D_+ J_-^{R,L} - D_- J_+^{R,L} = 0, \quad (2.6)$$

³The orthonormal coordinates $x^0 = t$ and $x^1 = x$ in two dimensions are related to the light-cone coordinates and derivatives as $x^\pm = \frac{1}{2}(t \pm x)$, and $\partial_\pm = \partial_t \pm \partial_x$.

⁴Our Lie algebra conventions are as follows. The anti-hermitian generators $\{T^a, a = 1, 2, \dots, n = \dim \mathfrak{g}\}$ of the Lie algebra \mathfrak{g} obey $[T^a, T^b] = f^{abc} T^c$ and $\text{Tr}(T^a T^b) = \delta^{ab}$. For any $X \in \mathfrak{g}$, $X = X^a T^a$.

and the superfield zero curvature condition is identically satisfied by J_{\pm} as

$$D_- J_+^{R,L} + D_+ J_-^{R,L} + i \left\{ J_+^{R,L}, J_-^{R,L} \right\} = 0, \quad (2.7)$$

We can expand the superfield $G(x^{\pm}, \theta^{\pm})$ as

$$G(x, \theta) = g(x) \left(1 + i\theta^+ \psi_+^R(x) + i\theta^- \psi_-^R(x) + i\theta^+ \theta^- F^R(x) \right), \quad (2.8)$$

An alternative component expansion is given as

$$G(x, \theta) = \left(i\theta^+ \psi_+^L(x) + i\theta^- \psi_-^L(x) + i\theta^+ \theta^- F^L(x) \right) g(x), \quad (2.9)$$

where ψ_{\pm} are the Majorana spinors such that

$$\psi_{\pm}^R = g^{-1} \psi_{\pm}^L g, \quad (2.10)$$

and $F(x)$ is the auxiliary field, with an algebraic equation of motion. The Majorana spinors $\psi_{\pm}(x)$ take values in the Lie algebra \mathfrak{g} of \mathcal{G} . The action of the symmetry $\mathcal{G}_L \times \mathcal{G}_R$ on component fields and the superfield currents $J_{\pm}^{R,L}$ is

$$\begin{aligned} g &\longmapsto UgV^{-1}, \\ \psi_{\pm}^R &\longmapsto V\psi_{\pm}^R V^{-1}, \\ \psi_{\pm}^L &\longmapsto U\psi_{\pm}^L U^{-1}, \\ J_{\pm}^R &\longmapsto VJ_{\pm}^R V^{-1}, \\ J_{\pm}^L &\longmapsto UJ_{\pm}^L U^{-1}, \end{aligned}$$

where U and V are the leading bosonic components of the matrix superfields \mathcal{U} and \mathcal{V} respectively, i.e., the fermions transform under \mathcal{G}_R . From now on we consider the spinors and current corresponding to \mathcal{G}_R , i.e., ψ_{\pm}^R and J_{\pm}^R (which we write as ψ_{\pm} and J_{\pm} during further discussion).

After the elimination of the auxiliary field from the expression of $G(x, \theta)$, the component Lagrangian

finally becomes⁵

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2} \text{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g \\
& + i \psi_+ \left(\partial_- \psi_+ + \frac{1}{2} [g^{-1} \partial_- g, \psi_+] \right) \\
& + \frac{i}{2} \psi_- \left(\partial_+ \psi_- + \frac{1}{2} [g^{-1} \partial_+ g, \psi_-] \right) \\
& + \frac{1}{2} \psi_+^2 \psi_-^2).
\end{aligned} \tag{2.11}$$

Using Euler-Lagrange equations, we can directly find the component equations of motion for the SPCM. From equation (2.5) we write the component expansion of superfield current of the SPCM as

$$\begin{aligned}
J_\pm = & \psi_\pm + \theta^\pm j_\pm - \frac{i}{2} \theta^\mp \{ \psi_+, \psi_- \} \\
& - i \theta^+ \theta^- \left(\partial_\pm \psi_\mp - [j_\pm, \psi_\mp] - \frac{i}{2} [\psi_\pm^2, \psi_\mp] \right),
\end{aligned} \tag{2.12}$$

where the components of the bosonic current are given by

$$j_\pm = - (g^{-1} \partial_\pm g + i \psi_\pm^2). \tag{2.13}$$

Again j_\pm represent the right bosonic current j_\pm^R . Substituting these into the superspace equations of motion, collecting terms and writing $h_\pm = \psi_\pm^2 \Leftrightarrow h_\pm^a = \frac{1}{2} f^{abc} \psi_\pm^b \psi_\pm^c$, we get the equations of motion for fermionic and bosonic fields of the SPCM,

$$\partial_\pm \psi_\mp - \frac{1}{2} [j_\pm, \psi_\mp] - \frac{i}{4} [h_\pm, \psi_\mp] = 0, \tag{2.14}$$

$$\partial_- j_+ + \partial_+ j_- = 0, \tag{2.15}$$

along with

⁵The two-dimensional Minkowski matrix is $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the γ -matrices $\gamma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The Dirac spinor is $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where ψ_\pm are chiral spinors and our assumption is that ψ_\pm are real (Majorana). The Lorentz behaviour of x^\pm, ∂_\pm and ψ_\pm is $x^\pm \mapsto e^{\mp\Lambda} x^\pm$, $\partial_\pm \mapsto e^{\mp\Lambda} \partial_\pm$ and $\psi_\pm \mapsto e^{\mp\frac{1}{2}\Lambda} \psi_\pm$, where Λ is the rapidity of the Lorentz boost. The rule of raising and lowering spinor indices is $\psi^\pm = \pm \psi_\mp$.

$$\partial_{\mp} j_{\pm} = -\frac{1}{2} [j_{\pm}, j_{\mp}] + \frac{i}{4} [j_{\mp}, h_{\pm}] - \frac{i}{4} [j_{\pm}, h_{\mp}] + \frac{1}{4} [h_{\pm}, h_{\mp}] . \quad (2.16)$$

We use the fermion equations of motion to get the following equations

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = i\partial_- h_+ - i\partial_+ h_-, \quad (2.17)$$

$$\partial_{\mp}(ih_{\pm}) = -\frac{1}{2} \left[ih_{\pm}, j_{\mp} + \frac{i}{2} h_{\mp} \right], \quad (2.18)$$

and

$$\partial_- h_+ + \partial_+ h_- = 0. \quad (2.19)$$

The equations (2.15) and (2.19) show the conservation of bosonic currents j_{\pm} and h_{\pm} respectively.

The Poisson brackets for the bosonic currents have already been derived in [2] and are given below

$$\begin{aligned} \{j_0^a(x), j_0^b(y)\} &= f^{abc} j_0^c(x) \delta(x-y), \\ \{j_0^a(x), j_1^b(y)\} &= f^{abc} j_1^c(x) \delta(x-y) + \delta^{ab} \delta'(x-y), \\ \{j_1^a(x), j_1^b(y)\} &= \frac{-i}{4} f^{abc} (h_+^c(x) + h_-^c(x)) \delta(x-y). \end{aligned} \quad (2.20)$$

In light-cone coordinates, the brackets are expressed as

$$\begin{aligned} \{j_{\pm}^a(x), j_{\pm}^b(y)\} &= \frac{1}{2} f^{abc} (3j_{\pm}^c(x) - j_{\mp}^c(x) - \frac{1}{2} i h_{\pm}^c(x) \\ &\quad - \frac{1}{2} i h_{\mp}^c(x)) \delta(x-y) + 2\delta^{ab} \delta'(x-y), \end{aligned} \quad (2.21)$$

$$\begin{aligned} \{j_+^a(x), j_-^b(y)\} &= \frac{1}{2} f^{abc} (j_+^c(x) - j_-^c(x) + \frac{1}{2} i h_+^c(x) \\ &\quad + \frac{1}{2} i h_-^c(x)) \delta(x-y), \end{aligned} \quad (2.22)$$

The fermions obey

$$\{\psi_{\pm}^a(x), \psi_{\pm}^b(y)\} = -i\delta^{ab}\delta(x-y), \quad (2.23)$$

$$\{\psi_+^a(x), \psi_-^b(y)\} = 0. \quad (2.24)$$

It is also useful to note that

$$\{h_{\pm}^a(x), \psi_{\pm}^b(y)\} = if^{abc}\psi_{\pm}^c(x)\delta(x-y), \quad (2.25)$$

$$\{h_{\pm}^a(x), h_{\pm}^b(y)\} = if^{abc}h_{\pm}^c(x)\delta(x-y). \quad (2.26)$$

We recall the definition of the standard Poisson structure associated with an arbitrary connected Lie group \mathcal{G} and consider a non-degenerate matrix d with entries [15], [16]

$$d^{ab} = \langle T^a, T^b \rangle. \quad (2.27)$$

where \langle, \rangle represents the killing form and for a semi-simple Lie algebra we have $\langle T^a, T^b \rangle = \delta^{ab}$. Since $Tr(T^a T^b) = \delta^{ab}$, therefore if \mathfrak{g} is a semi-simple and represented as a matrix algebra, we may assume $d^{ab} = Tr(T^a T^b)$. Let d_{ab} denote the entries of the inverse matrix d^{-1} , an element c of $\mathfrak{g} \otimes \mathfrak{g}$ and elements A^a of \mathfrak{g} be defined by

$$c = d_{ab}T^a \otimes T^b, \quad (2.28)$$

$$A_a = d_{ab}T^b. \quad (2.29)$$

Then we have the relations

$$[c, A_c \otimes I] = -[c, I \otimes A_c] = f^{abc}A_a \otimes A_b, \quad (2.30)$$

where the symbols $A \otimes I$ and $I \otimes A$ denote the natural embedding of A into $\mathfrak{g} \otimes \mathfrak{g}$.

Using the usual tensor product notation, the Poisson brackets can be expressed in the following way:

$$\{j_0(x) \otimes j_0(y)\} = [c, j_0(x) \otimes \mathbf{1}] \delta(x-y), \quad (2.31)$$

$$\begin{aligned} \{j_0(x) \otimes j_1(y)\} &= [c, j_1(x) \otimes \mathbf{1}] \delta(x-y) \\ &\quad + c\delta'(x-y), \end{aligned} \quad (2.32)$$

$$\{j_1(x) \otimes j_1(y)\} = -\frac{i}{4} [c, (h_+(x) + h_-(x)) \otimes \mathbf{1}] \delta(x-y), \quad (2.33)$$

$$\{j_1(x) \otimes h_\pm(y)\} = \pm \frac{1}{2} [c, h_\pm(x) \otimes \mathbf{1}] \delta(x-y), \quad (2.34)$$

$$\{j_0(x) \otimes h_\pm(y)\} = [c, h_\pm(x) \otimes \mathbf{1}] \delta(x-y), \quad (2.35)$$

$$\{h_\pm(x) \otimes h_\pm(y)\} = i [c, h_\pm(x) \otimes \mathbf{1}] \delta(x-y), \quad (2.36)$$

$$\{h_\pm(x) \otimes h_\mp(y)\} = 0. \quad (2.37)$$

In light-cone coordinates the above brackets can be expressed as

$$\begin{aligned} \{j_\pm(x) \otimes j_\pm(y)\} &= \frac{1}{2} [c, (3j_\pm(x) - j_\mp(x) \\ &\quad - \frac{1}{2}ih_+(x) - \frac{1}{2}ih_-(x)) \otimes \mathbf{1}] \delta(x-y) + 2c\delta'(x-y), \end{aligned} \quad (2.38)$$

$$\begin{aligned} \{j_+(x) \otimes j_-(y)\} &= \frac{1}{2} [c, (j_+(x) - j_-(x) \\ &\quad + \frac{1}{2}ih_+(x) + \frac{1}{2}ih_-(x)) \otimes \mathbf{1}] \delta(x-y), \end{aligned} \quad (2.39)$$

$$\{j_+(x) \otimes h_\pm(y)\} = \frac{3}{2} [c, (h_\pm(x) \otimes \mathbf{1}) \delta(x-y), \quad (2.40)$$

$$\{j_-(x) \otimes h_\pm(y)\} = \frac{1}{2} [c, (h_\pm(x) \otimes \mathbf{1}) \delta(x-y). \quad (2.41)$$

The SPCM is superconformally invariant classically, with the super energy-momentum tensor obeying

$$D_- \text{Tr}(J_+ J_+ +) = 0, \quad (2.42)$$

$$D_+ \text{Tr}(J_- J_- -) = 0, \quad (2.43)$$

where $J_{\pm \pm}$ is defined as

$$J_{\pm \pm} = D_\pm J_\pm + iJ_\pm^2. \quad (2.44)$$

The component content of the superspace conservation equations (2.42) and (2.43) correspond to the conservation of supersymmetry current and the energy-momentum tensor $T_{\pm \pm}$. The conservation equations

for $T_{\pm \pm}$ are

$$\partial_+ T_{- -} = 0,$$

$$\partial_- T_{+ +} = 0,$$

where the components $T_{\pm \pm}$ are given by

$$T_{\pm \pm} = \text{Tr} (i\psi_{\pm} \partial_{\pm} \psi_{\pm} + j_{\pm}^2 + ij_{\pm} \psi_{\pm}^2). \quad (2.45)$$

The Poisson brackets of the energy-momentum tensor components $T_{\pm \pm}$ are given as

$$\{T_{\pm \pm}(x), T_{\pm \pm}(y)\} = -8T_{\pm \pm} \delta'(x-y) - 4T'_{\pm \pm} \delta(x-y), \quad (2.46)$$

$$\{T_{+ +}(x), T_{- -}(y)\} = 0. \quad (2.47)$$

3 Lax pair and the extended Yang-Baxter relations

The field equations (2.14)-(2.15) of the SPCM Lagrangian (2.11) are also obtained as the compatibility condition of the following set of linear equations (Lax pair)[1]

$$\begin{aligned} \partial_+ V(x^+, x^-; \lambda) &= A_+^{(\lambda)} V(x^+, x^-; \lambda), \\ \partial_- V(x^+, x^-; \lambda) &= A_-^{(\lambda)} V(x^+, x^-; \lambda), \end{aligned} \quad (3.1)$$

where $A_{\pm}^{(\lambda)} = A_{\pm}(x^+, x^-; \lambda)$ is defined as,

$$A_{\pm}^{(\lambda)} = \left\{ \mp \left(\frac{\lambda}{1 \mp \lambda} \right) j_{\pm} + i \left(\frac{\lambda}{1 \mp \lambda} \right)^2 h_{\pm} \right\}. \quad (3.2)$$

The compatibility condition of the linear system (3.1) is the zero-curvature condition for the λ -dependent connection components $A_{\pm}^{(\lambda)}$

$$\begin{aligned} & \left[\partial_+ - A_+^{(\lambda)}, \partial_- - A_-^{(\lambda)} \right] \\ & \equiv \partial_- A_+^{(\lambda)} - \partial_+ A_-^{(\lambda)} + \left[A_+^{(\lambda)}, A_-^{(\lambda)} \right] = 0. \end{aligned} \quad (3.3)$$

Inserting from (3.2) in equation (3.3) gives

$$0 = -\left(\frac{\lambda}{1-\lambda}\right)\partial_- j_+ - \left(\frac{\lambda}{1+\lambda}\right)\partial_+ j_- + \frac{1}{2} \left(\frac{\lambda}{1-\lambda} - \frac{\lambda}{1+\lambda} \right) \times$$

$$\begin{aligned}
& \left([j_+, j_-] - \frac{i}{2} [j_-, h_+] + \frac{i}{2} [j_+, h_-] - \frac{1}{2} [h_+, h_-] \right) \\
& + \left(\frac{\lambda}{1-\lambda} \right)^2 \left(i\partial_- h_+ + \frac{1}{2} \left[ih_+, j_- + \frac{i}{2} h_- \right] \right) \\
& - \left(\frac{\lambda}{1+\lambda} \right)^2 \left(i\partial_+ h_- + \frac{1}{2} \left[ih_-, j_+ + \frac{i}{2} h_+ \right] \right).
\end{aligned} \tag{3.4}$$

Since equation (3.4) holds for all values of λ away from ± 1 , the coefficients of $\left(\frac{\lambda}{1-\lambda}\right)$, $\left(\frac{\lambda}{1+\lambda}\right)$, $\left(\frac{\lambda}{1-\lambda}\right)^2$ and $\left(\frac{\lambda}{1+\lambda}\right)^2$ must be separately zero. This gives equations (2.16) and (2.18) that are equivalent to the equations (2.14)-(2.15). The general solution of the Lax pair (3.1) is

$$V(x^+, x^-; \lambda) = e^{P(x^+, x^-; \lambda)} V_0(\lambda), \tag{3.5}$$

where

$$\begin{aligned}
P(x^+, x^-; \lambda) &= \frac{-\lambda}{1-\lambda} \int_{x_0^+}^{x^+} j_+ dy^+ + i \left(\frac{\lambda}{1-\lambda} \right)^2 \int_{x_0^+}^{x^+} h_+ dy^+ \\
&+ \frac{\lambda}{1+\lambda} \int_{x_0^-}^{x^-} j_- dy^- + i \left(\frac{\lambda}{1+\lambda} \right)^2 \int_{x_0^-}^{x^-} h_- dy^-.
\end{aligned}$$

In the above expression V_0 is the initial condition and is a free element of the Lie group \mathcal{G} . In terms of space-time coordinates, the associated linear system can be expressed as

$$\begin{aligned}
\partial_0 V(t, x; \lambda) &= A_0^{(\lambda)} V(t, x; \lambda), \\
\partial_1 V(t, x; \lambda) &= A_1^{(\lambda)} V(t, x; \lambda),
\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
A_0^{(\lambda)} &= \frac{-\lambda}{1-\lambda^2} \\
&\left\{ j_1 + \lambda j_0 - \frac{i}{2} \lambda \left(\frac{1+\lambda}{1-\lambda} \right) h_+ - \frac{i}{2} \lambda \left(\frac{1-\lambda}{1+\lambda} \right) h_- \right\},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
A_1^{(\lambda)} &= \frac{\lambda}{1-\lambda^2} \\
&\left\{ j_0 + \lambda j_1 - \frac{i}{2} \lambda \left(\frac{1+\lambda}{1-\lambda} \right) h_+ + \frac{i}{2} \lambda \left(\frac{1-\lambda}{1+\lambda} \right) h_- \right\}.
\end{aligned} \tag{3.8}$$

Using equations (2.31)-(2.37) to find the Poisson bracket of A_1 's (the spatial part of the Lax pair) from (3.8) we get

$$\begin{aligned} \{A_1(x, \lambda) \otimes A_1(y, \lambda)\} &= \left\{ \frac{-\lambda\mu}{(1-\lambda^2)(\lambda-\mu)} [c, A_1(x, \lambda) \otimes \mathbf{1}] \right. \\ &\quad + \frac{-\lambda\mu}{(1-\mu^2)(\lambda-\mu)} [c, \mathbf{1} \otimes A_1(x, \mu)] \left. \right\} \delta(x-y) \\ &\quad + \frac{\lambda\mu(\lambda+\mu)}{(1-\lambda^2)(1-\mu^2)} c \delta'(x-y). \end{aligned} \quad (3.9)$$

The terms containing the brackets of h_{\pm} cancel and we are left with the terms which can be written in terms of the Lax matrix $A_1(x, \lambda)$. In terms of r and s matrices we can rewrite the Poisson bracket as

$$\begin{aligned} \{A_1(x) \otimes A_1(y)\} &= \left\{ [(r-s)_{\lambda, \mu}, A_1(x, \lambda) \otimes \mathbf{1}] \right. \\ &\quad + \left. [(r+s)_{\lambda, \mu}, \mathbf{1} \otimes A_1(x, \mu)] \right\} \delta(x-y) \\ &\quad - 2s(\lambda, \mu) \delta'(x-y), \end{aligned} \quad (3.10)$$

This result is very important, which on comparison with the bosonic PCM [21] shows that the Poisson bracket is of the same form as that of the bosonic principal chiral model with no independent contribution coming from the terms containing the fields $h_{\pm}(x)$ and it reduces to the Poisson bracket of bosonic model found in [21], when fermions are set equal to zero. Since the Poisson bracket is same for both the bosonic PCM and SPCM, therefore the algebra of monodromy matrices obtained for the bosonic PCM [12] can be extended to the case of SPCM (see section 4). The matrices r and s are given as

$$r(\lambda, \mu) = \frac{-\lambda\mu}{2(\lambda-\mu)} \left\{ \frac{1}{1-\mu^2} + \frac{1}{1-\lambda^2} \right\} c \quad (3.11)$$

$$s(\lambda, \mu) = \frac{-\lambda\mu(\lambda+\mu)}{2(1-\mu^2)(1-\lambda^2)} c \quad (3.12)$$

Equations (3.11) and (3.12) show that the r and s matrices obtained for the SPCM are same as for the bosonic PCM. The antisymmetry of the canonical brackets (3.10) holds through the relations

$$Pr(\lambda, \mu)P = -r(\mu, \lambda), \quad Ps(\lambda, \mu)P = s(\mu, \lambda), \quad (3.13)$$

where $P_{ac, bd} = \delta_{ad}\delta_{cb}$ satisfies for any matrices A, B ; $P(A \otimes B)P = B \otimes A$. It must be emphasized that the algebra (3.10) is a non-trivial generalization of the canonical structure of those ultralocal type models which are obtained in the limit $s = 0$ and $\partial_x r(x, \lambda, \mu) = 0$. Our algebra (3.10) is a linear algebra written

in terms of two matrix structure constants (r and s) with central extension (δ' term) governed by the s -matrix. It is important to point out here that in general the non-ultralocal integrable models exhibit a space-time dependence for r and s matrices [18]-[21] and there could be higher derivatives of the delta function in the Poisson current algebra. In our case, the supersymmetric model is of non-ultralocal type containing first derivative of the delta function in its Poisson current algebra. However, the r and s matrices do not contain space-time dependence and therefore are non-dynamical. Such kind of non-dynamical $r-s$ matrices also appear in the case of an $SU(2)$ WZW model as a non-ultralocal model [12]. The algebra can be expressed in a more transparent way by introducing Lax operator $\mathcal{D}(x, \lambda)$, defined as

$$\mathcal{D}_1(x, \lambda) = \partial_1 + A_1(x, \lambda), \quad (3.14)$$

so that the Poisson bracket algebra can be equivalently expressed in terms of the differential operator $\mathcal{D}(x, \lambda)$ as

$$\begin{aligned} \{\mathcal{D}_1(x, \lambda) \otimes \mathcal{D}_1(y, \mu)\} &= -[r(x, \lambda, \mu) \delta(x - y), \mathcal{D}_1(x, \lambda) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{D}_1(y, \mu)] \\ &\quad + [s(x, \lambda, \mu) \delta(x - y), \mathcal{D}_1(x, \lambda) \otimes \mathbf{1} \\ &\quad - \mathbf{1} \otimes \mathcal{D}_1(y, \mu)]. \end{aligned} \quad (3.15)$$

Requiring now, the Jacobi identity of the canonical bracket (3.10) to be satisfied, we get the following, extended Yang-Baxter equation for the numerical r - and s -matrices (see [17])

$$\begin{aligned} &[(r + s)_{13}(\lambda, \eta), (r - s)_{12}(\lambda, \mu)] \\ &+ [(r + s)_{23}(\mu, \eta), (r + s)_{12}(\lambda, \mu)] \\ &+ [(r + s)_{23}(\mu, \eta), (r + s)_{13}(\lambda, \eta)] \\ &= 0. \end{aligned} \quad (3.16)$$

Here the indices 1, 2, 3 label the three spaces involved in computing the algebra of three A_1 -matrices and we have for example $(r + s)_{12}(\lambda, \mu) = (r + s)(\lambda, \mu) \otimes \mathbf{1}_3$. Again, (3.16) is a generalization of the usual classical Yang-Baxter equation for r -matrices in ultralocal type models which is obtained (in 3.16 as in 3.10) for $s = 0$. However, we have to note that in going from ultralocal type models to non-ultralocal

ones, it is not sufficient to simply add a central extension (δ' term) to the ultralocal algebra of A_1 -matrices. In fact, it is necessary, as can be seen from (3.10), to modify also the $\delta(x-y)$ part with s terms related to the extension (δ' term) in order to satisfy the Jacobi identity (3.16). Moreover, in general (3.16) holds with an r -matrix which itself does not satisfy the usual classical Yang-Baxter equation for r -matrices of ultralocal models, hence, showing the crucial role played by the new s -matrix. It is then possible, using (3.10) and (3.16) to derive the canonical algebra of two monodromy matrices in a completely consistent manner, i.e., in agreement with the Jacobi identity since (3.16) is verified.

4 Algebra of monodromy matrices

The monodromy matrix $T(x, y, \lambda)$ is defined in terms of Lax matrix $A_1(x, \lambda)$ as

$$T(x, y, \lambda) = P \exp \int_y^x A_1(x', \lambda) dx'. \quad (4.1)$$

The infinite volume limit of $T(x, y, \lambda)$ i.e.,

$$T(\infty, -\infty, \lambda) \equiv T(\lambda) = P \exp \int_{-\infty}^{\infty} A_1(x, \lambda) dx, \quad (4.2)$$

is a conserved quantity for any value of spectral parameter λ [20]. By expanding $T(\lambda)$ in powers of λ , an infinite set of non-local conserved quantities is obtained with the first two quantities given by [1]-[7].

$$Q^{(1)a} = - \int_{-\infty}^{\infty} dy \, j_0^a(t, y), \quad (4.3)$$

$$\begin{aligned} Q^{(0)a} = & \int_{-\infty}^{\infty} dy \, [-j_1^a(t, y) + \frac{i}{2} (h_+^a(t, y) - h_-^a(t, y)) \\ & + \frac{1}{2} f^{abc} j_0^b(t, y) \int_{-\infty}^y dz \, j_0^c(t, z)]. \end{aligned} \quad (4.4)$$

The non-local conserved quantities generate a Yangian deformation symmetry [6].

The monodromy matrix usually contains the main information about the canonical structure of the non-ultralocal sigma models. In particular, its infinite volume limit (through proper regularization) provides us when expanded in a power series in λ , with an infinite set of conserved quantities, an infinite subset of them being in involution i.e., Poisson commute, as a signature of complete integrability of the model. Since the Poisson bracket algebra of the A_1 -matrices of the SPCM is similar to that of the bosonic model, therefore the Poisson bracket for the monodromy matrices of the SPCM can be determined using the equal point limits through a regularization procedure developed for the bosonic models in [11]-[23]. The Poisson bracket of the monodromy matrices of the SPCM turns out to be of the same form as the Poisson bracket of the bosonic models and is given by

$$\{T(x, y, \lambda) \otimes T(x, y, \mu)\} = [r(\lambda, \mu), T(x, y, \lambda) T(x, y, \mu)]. \quad (4.5)$$

In the infinite volume limit equation (4.5) reads

$$\{T(\lambda) \otimes T(\mu)\} = [r(\lambda, \mu), T(\lambda) T(\mu)], \quad (4.6)$$

and the conserved quantities, $\text{Tr} T(\lambda)$ are in involution, being

$$\text{Tr}(A \otimes B) = \text{Tr} A \cdot \text{Tr} B, \quad (4.7)$$

so that

$$\{\text{Tr} T(\lambda), \text{Tr} T(\mu)\} = \text{Tr} \{T(\lambda) \otimes T(\mu)\} = 0. \quad (4.8)$$

In summary, we have calculated the Poisson bracket algebra of the A_1 -matrices of the Lax pair of the SPCM as a non-ultralocal integrable model. From the A_1 -matrices of the SPCM, we have determined the Poisson bracket algebra of the monodromy matrices and using the equal-point limit, we have shown the existence of conserved quantities of the model that are in involution with each other establishing the classical integrability of the SPCM as a non-ultralocal model. It seems appropriate here to make few comments about the existence of an infinite number of conserved quantities of the SPCM. It has been shown in [1], [2] that there exist an infinite number of nonlocal and local conserved quantities of the SPCM. The nonlocal conserved quantities can be generated from the monodromy matrix as has been discussed in this section and they generate Yangian symmetry. The local conserved quantities of the SPCM have been investigated

in [2] and it has been shown that there are two families of conserved quantities in involution, each with finitely many members whose spins are the exponents of the underlying Lie algebra. Similarly in [1], an infinite number of local conservation laws has been constructed through a pair of matrix Riccati equations of the SPCM. The appearance of these conserved quantities has important consequences regarding the integrability of the SPCM and they are constructed through a Lax pair and the zero-curvature condition of the model. No explicit form of the conserved quantities has been obtained either those of the trace of monodromy matrix or those obtained through a set of matrix Riccati equations. Once the explicit form of the conserved quantities is known, one would be able to establish a relation among these quantities generated through different approaches.

5 Conclusions

We have developed an $r - s$ matrix formalism of the supersymmetric principal chiral model as a non-ultralocal integrable model. By evaluating the fundamental Poisson bracket of the A_1 -matrices of the Lax pair of the SPCM, we have shown that this bracket has the same form as the fundamental Poisson bracket of the bosonic principal chiral model. The fundamental Poisson bracket is then used to define the monodromy matrix of the model, that gives the conserved quantities in involution. The algebraic structures studied here can also be investigated for the supersymmetric nonlinear sigma models on Riemannian symmetric spaces that is the most general class of supersymmetric nonlinear sigma models to be integrable (see e.g. [8], [9]). The other direction where the work can be further extended is the recent investigations regarding the classical integrability in superstring theory on the $AdS_5 \times S^5$ (see e.g. [24]-[28]). In these studies, the theory has been regarded as a nonlinear sigma model with the field taking values in the supercoset space $\frac{PSU(2,2/4)}{SO(4,1) \times SO(5)}$, which has an even part the $AdS_5 \times S^5$ geometry. The even part admits a Lax formalism and is further linked with conserved quantities of the Yang-Mills sector of the AdS/CFT correspondence (see e.g. [24]-[28]). The algebra of monodromy matrices for the $AdS_5 \times S^5$ superstrings has been investigated in [24]-[28]. In the light of our result, one can expect that Poisson bracket algebra can be developed for the superstring theory on $AdS_5 \times S^5$ as a non-ultralocal theory that gives conserved quantities in involution and it fits in the $r - s$ matrix formalism of integrable models. Another important direction that can be pursued for future research is to develop an $r - s$ matrix formalism for the sigma models with target space

supersymmetry. The most important aspect of such investigations is, however, to promote the classical integrability of such model to the quantum level.

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